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Floquet Spectrum for Two-Level Systems in Quasiperiodic Time-Dependent Fields

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We study the time evolution of N-level quantum systems under quasiperiodic time-dependent perturbations. The problem is formulated in terms of the spectral properties of a quasienergy operator defined in an enlarged Hilbert space, or equivalently of a generalized Floquet operator. We discuss criteria for the appearance of pure point as well as continuous spectrum, corresponding respectively to stable quasiperiodic dynamics and to unstable chaotic behavior. We discuss two types of mechanisms that lead to instability. The first one is due to near resonances, while the second one is of topological nature and can be present for arbitrary ratios between the frequencies of the perturbation. We treat explicitly an example of this type. The stability of the pure point spectrum under small perturbations is proven using KAM techniques.

KEY WORDS: Quasiperiodic; Floquet operator; quasienergy; quantum chaos; KAM; Rabi oscillations.

1. INTRODUCTION

It is a well-known fact that the time evolution of an isolated quantum system, described by a Hamiltonian H_0 with a discrete spectrum, cannot exhibit the type of behavior usually associated with deterministic chaos of classical systems. This follows from the fact that, independently of the nature of the dynamics generated by the corresponding classical Hamiltonian (when there is one), the quantum time evolution of the state

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$\varphi(t)$ is almost-periodic, since it can be expanded in terms of the eigenfunctions φ_n of H_0, with eigenvalues E_n:

$$\varphi(t)=\sum_n c_n e^{-iE_n t}\varphi_n \tag{1.1}$$

As a consequence, quantities like the expectation value of the energy $\langle \varphi(t)\, H_0 \varphi(t)\rangle$ or the correlation function

$$S_\varphi(t)=\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau ds\ \langle \varphi(t+s), \varphi(s)\rangle \tag{1.2}$$

are also almost-periodic.

Of course when the spacing between the levels becomes very small, as is always the case when we deal with macroscopic systems, the quantum evolution can, and generally will, imitate various features of classical behavior to any desired degree of accuracy. Interestingly enough, however, even very small quantum systems such as atoms and molecules can exhibit nontrivial dynamical behavior whenever they are subjected to a time-dependent external perturbation. The general problem is then the following: Suppose we subject the system described by H_0 to a time-dependent field; does the peturbed system have a behavior that is qualitatively similar to the unperturbed one? In particular: is the evolution still almost-periodic or it does it have a more complicated chaotic behavior? In the latter case one can investigate properties like the rate of decay of correlations or whether quantities like the energy stay bounded.

This question of stability has been studied for different kinds of time-dependent forces, from periodic[1–5] to fully stochastic,[6] passing through quasiperiodic[7–12] and deterministic forces with varying degrees of ergodicity.[13,14]

In the case of periodic forces the question of stability can be formulated in terms of the spectral properties of the quasienergy operator[15–19] or equivalently of the Floquet operator. Bellissard[20] proposed a generalization of the quasienergy operator to a large class of time-dependent forces. Some aspects of this approach were studied in ref. 21 for the case of quasiperiodic forces, where a generalized Floquet operator was introduced whose spectral properties are equivalent to the ones of the quasienergy operator. It was noted in ref. 21 that there are important qualitative differences between the periodic and quasiperiodic cases. These are manifested particularly clearly when the Hilbert space $\mathscr{H}$ is finite-dimensional, e.g., a spin-1/2 system in a variable magnetic field, or an idealized atom (where one can restrict oneself to a finite number of levels) in a radiation field. In the periodic case the Floquet operator is a finite-dimensional unitary matrix, and thus the spectrum of the quasienergy operator is always pure point. This need not be so for quasiperiodic fields. Numerical studies with a field containing two incommensurate frequencies reported in refs. 8 and 10 show an apparently continuous power spectrum leading to chaotic behavior with asymptotic decay of correlations. However, the analysis of this system by different methods[9] involving double Poincaré sections indicates that its evolution is regular (almost-periodic with very long quasiperiods). On the other hand, an example was constructed in ref. 11, with the time dependence of the field based on the Fibbonacci sequence, for which one can explicitly show that generically the time evolution is not almost-periodic. In the present work we give a qualitative interpretation of the simulations of refs. 8–10 and then carry out some rigorous analysis on conditions for regular and disordered behavior under smooth quasiperiodic time-dependent perturbations. We will discuss two different types of mechanism: the first one involves phenomena that are close to resonances, i.e., they appear when the ratio of the frequencies is well approximated by rationals. The second mechanism is of topological nature and can appear for arbitrary frequencies. It generally becomes observable only when the intensity of the perturbation is larger than some critical value. Physical examples of such instabilities occur, e.g., in microwave ionizations of Rydberg atoms[22] or of electrons attracted to a surface of liquid helium.[23] The close to resonant mechanism appears already in the simplest case in which $\mathscr{H}$ is one-dimensional. By diagonal composition one obtains also examples for the N-level models. For the second kind one needs at least two degrees of freedom, and one requires nonperturbative methods.

In Section 2 we discuss the example due to Pomeau *et al.*[8] and give an explanation of the apparently chaotic behavior, which, however, appears to be regular under the double Poincaré section analysis of ref. 9.

In Section 3 we introduce the notation and give some general results for quasiperiodic forces. In Section 4 we discuss, for the scalar case $\mathscr{H} = \mathbb{C}$, criteria for point spectrum and for continuous spectrum arising from a close to resonance mechanism. In Section 5 we discuss as an example a two-level system ($\mathscr{H} = \mathbb{C}^2$) which has absolutely continuous spectrum for any ratio of frequencies that is produced by a topological mechanism. In Section 6 we state a stability result or small perturbations. The proof involves a small-denominator problem that is treated with a KAM-type algorithm.

2. INTERPRETATION OF NUMERICAL STUDIES

A particularly simple example, which was treated numerically in refs. 8 and 10, is a two-level system with Hamiltonian

$$H = v\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + gF(t)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \tag{2.1}$$

with $F(t) = \cos(\omega_1 t)\cos(\omega_2 t)$, $v = 1/2$, $\omega_1 = 17711/28657$, $\omega_2 = 4637/13313$, acting on vectors $(\psi^{(1)}, \psi^{(2)})$. For $g = 5$ one observes a power spectrum that looks quite broad-banded and is interpreted as signature of chaotic behavior. On the other hand, the analysis of the same model performed by different methods involving double Poincaré sections indicates that the time evolution is regular,[9] quasiperiodic with a long quasiperiod. We propose the following interpretation of these numerical data: The relevant parameter of the problem is the quotient g/v. The two limiting cases $g = 0$ and $v = 0$ can be solved explicitly and have regular quasiperiodic solutions. For $g = 0$ the time evolution operator is

$$U(t, s) = \begin{pmatrix} e^{-iv(t-s)} & 0\\ 0 & e^{iv(t-s)}\end{pmatrix} \tag{2.2}$$

For $v = 0$ the Hamiltonian can be diagonalized by the unitary transformation

$$R = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

We have

$$R^{\dagger}HR = gF(t)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{2.3}$$

and therefore

$$\begin{aligned} U(t, 0) &= R\begin{pmatrix} e^{-igG(t)} & 0\\ 0 & e^{+igG(t)}\end{pmatrix} R^{\dagger}\\ G(t) &= \int_0^t dt'\, F(t') = \frac{\sin(\omega_1+\omega_2)t}{2(\omega_1+\omega_2)} + \frac{\sin(\omega_1-\omega_2)t}{2(\omega_1-\omega_2)} \end{aligned} \tag{2.4}$$

From this expression we see that the evolution for $v = 0$ is certainly quasiperiodic, but if g is large, the quasiperiods can be very long. The quasiperiodicity implies that $U(t, 0)$ can be represented by a series of the form

$$U(t, 0) = \sum_{n_1, n_2} \bar{U}_{n_1, n_2}\, e^{i(n_1\omega_1 + n_2\omega_2)t}$$

i.e., its Fourier spectrum consists of a set of frequencies that is dense. For small g only few of these frequencies appear with a relevant weight and the dynamics is simple. As g increases, more frequencies have important weight and it becomes very hard to distinguish numerically the dense Fourier spectrum from a continuous one. Therefore if a correlation function or power spectrum is computed using a time series that does not cover a long enough time interval, it can look chaotic. This is illustrated in Fig. 1, which shows the power spectrum corresponding to the observable

$$A(t) \equiv |\psi^{(2)}(t)|^2 - |\psi^{(1)}(t)|^2 = A(0)\cos[2gG(t)] \tag{2.5}$$

computed using a time series of 2^{15} points and a maximal t equal to 8000. This power spectrum is qualitatively very similar to the one computed in ref. 8 for the same parameters and the full Hamiltonian.

A perturbation analysis like the one described in Section 6 provides the result that for fixed v the dynamics will be almost-periodic for small g but also for very large g (since the limit $g \to \infty$ is equivalent to the limit $v \to 0$). From these arguments we conclude that the apparently irregular behavior observed in refs. 8 and 10 corresponds to the asymptotic regime of $g \to \infty$, which is complicated but quasiperiodic nonetheless in accordance with ref. 9.

This example shows that the interpretation of numerical simulations of these models is a quite subtle problem. In what follows we will discuss analytical methods that provide some criteria for regular as well as for irregular behavior, and we will decribe a model for which one can show that the dynamics is chaotic.

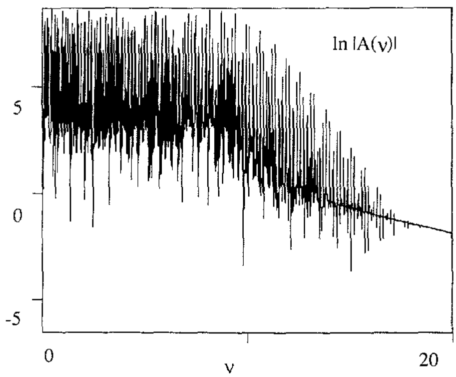

Fig. 1. Power spectrum corresponding to $A(t) = |\psi^{(2)}(t)|^2 - |\psi^{(1)}(t)|^2$ for $g = 5$ computed from a time series of 2^{15} points and maximal t equal to 8000.

3. GENERAL FORMALISM

We consider time-dependent Hamiltonians of the type

$$H = H(\boldsymbol{\theta}(t)) = H_0(x) + V(x; \boldsymbol{\theta}(t)) \tag{3.1}$$

on a Hilbert space $\mathscr{H}$, where x are the variables of the system and the force $\boldsymbol{\theta}(t) = g_t\boldsymbol{\theta} \in \mathscr{M}$ is described by an invertible flow on a compact manifold $\mathscr{M}$ with an ergodic invariant measure μ. Under some regularity conditions on V there is a unitary time evolution operator $U(t, s; \boldsymbol{\theta})$. The generalized quasienergy operator (QEO) K is defined[20] on an enlarged space $\mathscr{K} = \mathscr{H} \otimes L_2(\mathscr{M}, d\mu)$ by

$$[e^{-iKt}\Psi](\boldsymbol{\theta}) \equiv U(0, -t; \boldsymbol{\theta})\, \mathscr{T}_{-t}\Psi(\boldsymbol{\theta}) \equiv \mathscr{T}_{-t} U(t, 0; \boldsymbol{\theta})\, \Psi(\boldsymbol{\theta}) \tag{3.2}$$

where $[\mathscr{T}_{-t}\Psi](\boldsymbol{\theta}) = \Psi(g_{-t}\boldsymbol{\theta})$. The operator K acts as

$$[K\Psi](\boldsymbol{\theta}) = -i\frac{d}{dt}\Psi(g_t\boldsymbol{\theta})\bigg|_{t=0} + H(\boldsymbol{\theta})\,\Psi \tag{3.3}$$

In the case of a periodic force: $\mathscr{M} = S^1$ is the unit circle, $g_t\theta = \theta + \omega t$, $d\mu = d\theta$, and $K = -i\omega\, \partial/\partial\theta + H(\theta)$.

Here we will discuss the case of quasiperiodic forces with two incommensurate frequencies: $\omega_1/\omega_2 \equiv \alpha \notin \mathbb{Q}$. The manifold $\mathscr{M}$ is a torus $S^1 \times S^1$, the flow $g_t(\theta_1, \theta_2) = (\omega_1 t + \theta_1, \omega_2 t + \theta_2)$, $d\mu = d\theta_1\, d\theta_2$, and

$$K = -i\omega_1 \frac{\partial}{\partial\theta_1} - i\omega_2 \frac{\partial}{\partial\theta_2} + H(\theta_1, \theta_2) \tag{3.4}$$

We denote the two periods by $T_j = 2\pi/\omega_j$. The relation (3.2) provides a link between the spectral properties of K and the stability of the dynamics of the system. As discussed in refs. 20, 21, and 24, point spectrum implies almost-periodic evolution, while continuous spectrum signals an instability with a more complicated behavior and decaying correlations.

It is useful to introduce a generalized Floquet operator[21] whose spectral properties are equivalent to those of the quasienergy operator. It acts on $\mathscr{K}_1 \equiv \mathscr{H} \otimes L_2(S^1, d\theta_1)$ and is defined by

$$U_{\mathrm{F}} = \mathscr{T}^1_{-T_2} u_1(\theta_1) \tag{3.5}$$

where $u_1(\theta_1) \equiv U(T_2, 0; \theta_1, 0)$ ($\equiv$monodromy operator) and $[\mathscr{T}^1_{-T_2}\phi](\theta_1) = \phi(\theta_1 - \omega_1 T_2)$. The following result was shown in ref. 21.

Lemma 3.1. (i) If $\phi \in \mathscr{K}_1$ is an eigenfunction of

$$U_{\mathrm{F}}\phi = e^{-i\lambda T_2}\phi \tag{3.6}$$

then the function ψ defined by

$$\psi(\theta_1, \theta_2) = e^{i\lambda\theta_2/\omega_2} U(0, -\theta_2/\omega_2; \theta_1, \theta_2)\, \phi(\theta_1 - \theta_2\omega_1/\omega_2) \tag{3.7}$$

belongs to $\mathscr{K}$ and is an eigenfunction of the QEO, with eigenvalue λ:

$$K\psi = \lambda\psi \tag{3.8}$$

(ii) Conversely, if $\psi \in \mathscr{K}$ is an eigenfunction of the QEO, then there is a function $\phi \in \mathscr{K}_1$ such that ψ can be represented by (3.7), and ϕ is an eigenfunction of the generalized Floquet operator.

Remark. The eigenvalue equation for the Floquet operator (3.6) can be written as

$$u_1(\theta_1)\, \phi(\theta_1) = e^{-i\lambda T_2} \phi(\theta_1 + 2\pi\alpha) \tag{3.9}$$

which has the form of a cohomology equation.

Before we discuss some examples, we give some general properties of the eigenfunctions of the Floquet operator, which are mostly a consequence of the ergodicity of the dynamical system on the torus $\mathscr{M}$.

Lemma 3.2. If $\phi_1, \phi_2 \in \mathscr{K}_1 \equiv \mathbb{C}^N \otimes L_2(S^1, d\theta_1)$ are eigenfunctions of the Floquet operator U_F, i.e.,

$$u_1(\theta_1)\, \phi_k(\theta_1) = \Lambda_k \phi_k(\theta_1 + 2\pi\alpha) \tag{3.10}$$

then the absolute value of the $\mathscr{H}$-scalar product

$$|\langle \phi_1(\theta_1), \phi_2(\theta_1) \rangle_{\mathscr{H}}| \equiv \left| \sum_{j=1}^{N} \phi_1^{(j)*}(\theta_1)\, \phi_2^{(j)}(\theta_1) \right|$$

is constant for almost all $\theta_1 \in S^1$.

Proof. Since U_F is unitary, $|\Lambda_k| = 1$. The function defined by $g(\theta_1) = |\langle \phi_1(\theta_1), \phi_2(\theta_1) \rangle_{\mathscr{H}}|$ is in $L_1(S^1, d\theta_1)$ and is invariant under the map $\theta_1 \mapsto \theta_1 + 2\pi\alpha$ on S^1:

$$\begin{aligned} g(\theta_1 + 2\pi\alpha) &= |\langle \phi_1(\theta_1 + 2\pi\alpha), \phi_2(\theta_1 + 2\pi\alpha) \rangle_{\mathscr{H}}| \\ &= \left| \left\langle \frac{1}{\Lambda_1} u_1(\theta_1)\, \phi_1(\theta_1), \frac{1}{\Lambda_2} u_1(\theta_1)\, \phi_2(\theta_1) \right\rangle_{\mathscr{H}} \right| \\ &= \frac{1}{|\Lambda_1|\, |\Lambda_2|}\, g(\theta_1) = g(\theta_1) \end{aligned} \tag{3.11}$$

Therefore, since the map is ergodic, g is constant for a.a. θ_1.[25]

Corollary 3.3. The eigenfunctions of U_F have $\mathscr{H}$-norm $\|\phi(\theta_1)\|_{\mathscr{H}}$ that is constant for almost all $\theta_1 \in S^1$.

Corollary 3.4. For the case $\mathscr{H} = \mathbb{C}^N$ with $N = 1, 2$, if U_F has one eigenfunction, then the whole spectrum is pure point.

Proof. (i) $N = 1$: If ϕ is an eigenfunction with eigenvalue Λ, then $\phi_m = e^{im\theta_1}\phi$, $m \in \mathbb{Z}$, are also eigenfunctions with eigenvalue $\Lambda e^{-im2\pi\alpha}$. We will show that the $\{\phi_m\}$ form a complete basis of $\mathscr{K}_1 = L_2(S^1, d\theta_1)$: The set $\{e^{im\theta_1}\}$ is a basis and, by Corollary 3.3, ϕ can be chosen such that $|\phi(\theta_1)| = 1$ for all θ_1, and thus $1/\phi$ is well defined. Therefore any function $f \in \mathscr{K}_1$ can be expanded as

$$f(\theta_1) = \sum_m a_m \phi_m(\theta_1) \tag{3.12}$$

where a_m is the Fourier coefficient of f/ϕ:

$$a_m = \frac{1}{2\pi} \int_{S^1} d\theta_1 \frac{f(\theta_1)}{\phi(\theta_1)} e^{-im\theta_1} \tag{3.13}$$

(ii) $N = 2$: We generalize the preceding argument as follows: First remark that if $\phi_1 = (y(\theta_1), z(\theta_1))$ is an eigenfunction with eigenvalue Λ, then we get a second one with eigenvalue Λ^* of the form $\phi_2 = (-z(\theta_1), y(\theta_1))$. This can be verified by direct insertion into the eigenvalue equation. According to Corollary 3.3, the $\mathscr{H}$-norm of the eigenfunctions is constant almost everywhere; therefore we can choose y and z such that $|y(\theta_1)|^2 + |z(\theta_1)|^2 = 1$ for all θ_1. We will prove that the set $\{\phi_1 e^{in\theta_1}, \phi_2 e^{in\theta_1}\}_{n\in\mathbb{Z}}$ is a basis of $\mathscr{K}_1$, i.e., that any function $(\mu(\theta_1), \nu(\theta_1)) \in \mathscr{K}_1$ can be represented as a linear combination

$$\begin{pmatrix} \mu(\theta_1) \\ \nu(\theta_1) \end{pmatrix} = \sum_{n\in\mathbb{Z}} a_n \phi_1 e^{in\theta_1} + b_n \phi_2 e^{in\theta_1}; \qquad a_n, b_n \in \mathbb{C} \tag{3.14}$$

We construct a 2×2 matrix R that has the eigenfunctions ϕ_1 and ϕ_2 in the columns, and $R^\dagger$ is its conjugate

$$R = \begin{pmatrix} y & -z^* \\ z & y^* \end{pmatrix}, \qquad R^\dagger = \begin{pmatrix} y^* & z^* \\ -z & y \end{pmatrix} \tag{3.15}$$

Viewed as operators in $\mathscr{K}_1$, they satisfy $RR^\dagger = \mathbf{1} = R^\dagger R$. Equation (3.14) can be equivalently written as

$$\begin{pmatrix} \mu(\theta_1) \\ \nu(\theta_1) \end{pmatrix} = \sum_{n\in\mathbb{Z}} e^{in\theta_1} R(\theta_1) \begin{pmatrix} a_n \\ b_n \end{pmatrix} \tag{3.16}$$

If we define a_n, b_n as

$$\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \frac{1}{2\pi} \int_{S^1} d\theta'_1 e^{-in\theta'_1} R^\dagger(\theta'_1) \begin{pmatrix} \mu(\theta'_1 \\ \nu(\theta'_1) \end{pmatrix} \tag{3.17}$$

we verify by insertion that (3.16) is identically satisfied:

$$\begin{aligned} \begin{pmatrix} \mu(\theta_1) \\ \nu(\theta_1) \end{pmatrix} &= \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \int_{S^1} d\theta'_1 e^{in(\theta_1 - \theta'_1)} R(\theta_1)\, R^\dagger(\theta'_1) \begin{pmatrix} \mu(\theta'_1) \\ \nu(\theta'_1) \end{pmatrix} \\ &= \int_{S^1} d\theta'_1\, \delta(\theta_1 - \theta'_1)\, R(\theta_1)\, R^\dagger(\theta'_1) \begin{pmatrix} \mu(\theta'_1) \\ \nu(\theta'_1) \end{pmatrix} = \begin{pmatrix} \mu(\theta_1) \\ \nu(\theta_1) \end{pmatrix} \end{aligned} \tag{3.18}$$

Remark. This argument is readily generalized to the case $\mathscr{H} = \mathbb{C}^N$. The key point is that in (3.18) we need $R(\theta_1)\, R^\dagger(\theta_1)$ to be equal to the identity in $\mathscr{H}$ for almost all θ_1. This can be achieved by means of Lemma 3.2 by constructing R with N eigenvectors ϕ_j that are mutually orthogonal in $\mathscr{H}$ for almost all θ_1. Also, if one knows $N-1$ such ϕ_j, an Nth one can be constructed by orthogonalization.

This gives us an insight about the composition of the set of eigenfunctions of U_F. For $\mathscr{H} = \mathbb{C}^N$ the set of eigenfunctions has the structure $\{e^{in\theta_1}\phi_j(\theta_1)\}$, $n \in \mathbb{Z}$, $j = 1,..., N$, where ϕ_j are N functions that are mutually orthogonal in $\mathscr{H}$ for almost all θ_1 (and therefore form a basis of $\mathscr{H}$ for a.a. θ_1). The corresponding eigenvalues are of the form $\exp[-i(\lambda_j T_2 + n_1 2\pi\alpha)]$, and thus they form a dense set.

Remark. In the case $\mathscr{H} = \mathbb{C}^2$, if we consider Hamiltonians with zero trace (see beginning of Section 5) the propagator at any fixed time is an element of the group $SU(2)$. If we define $u_k(\theta_1) = U(kT_2, 0; \theta_1, 0)$, $k \in \mathbb{Z}$, we get a family of maps from the circle to $SU(2)$ that is a cocycle, i.e., it satisfies the following condition[26,27]:

$$u_k(\theta_1) = u_1(\theta_1 + (k-1)\, 2\pi\alpha) \cdots u_1(\theta_1 + 2\pi\alpha)\, u_1(\theta_1) \tag{3.19}$$

i.e., if $u_1(\theta_1)$ is given for all $\theta_1 \in S^1$, the propagator $U(t, 0; \theta_1, 0)$ is determined for all times that are integer multiples of T_2. The relation (3.19) follows immediately from the identity

$$U(t + a, s + a; \theta_1, \theta_2) = U(t, s; \theta_1 + \omega_1 a, \theta_2 + \omega_2 a) \tag{3.20}$$

Two cocycles $u_k(\theta_1)$, $u'_k(\theta_1)$ are called cohomologous to each other if there is a map $R: S^1 \to SU(2)$ such that

$$R^{-1}(\theta_1 + 2\pi\alpha)\, u_k(\theta_1)\, R(\theta_1) = u'_k(\theta_1) \tag{3.21}$$

From the proof of Corollary 3.4 it follows that the statement that the Floquet operator has pure point spectrum is equivalent to the statement that the cocycle $u_k(\theta_1)$ is cohomologous to a constant cocycle (i.e., one that does not depend on θ_1 and thus can be trivially diagonalized). To see this, we define the diagonal matrix $D = \mathrm{diag}(\Lambda, \Lambda^*)$. Taking R as defined in (3.15), the eigenvalue equation for the Floquet operator can be written as

$$u_k(\theta_1)\, R(\theta_1) = R(\theta_1 + 2\pi\alpha)\, D \tag{3.22}$$

which is identical to the cohomology equation (3.21), where $u'_k(\theta_1) = D$ is a constant cocycle.

4. SCALAR EXAMPLES WITH CONTINUOUS SPECTRUM

The special case

$$H = h_3(t)\, \sigma_3 = h_3(t) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.1}$$

can be reduced to the solution of a scalar problem. We will show that already in this simplest case it is possible to have continuous quasienergy spectrum. In the scalar case the Hamiltonian is a function

$$H = f(\omega_1 t + \theta_1, \omega_2 t + \theta_2) \tag{4.2}$$

that acts multiplicatively on $\mathscr{H} = \mathbb{C}$. The propagator can be written explicitly as

$$U(t, s; \boldsymbol{\theta}) = \exp\left[-i \int_s^t dt'\, f(\omega_1 t' + \theta_1, \omega_2 t' + \theta_2) \right] \tag{4.3}$$

The question of stability reduces thus to the question of whether the integral of a quasiperiodic function is almost-periodic.[28,29] The Floquet operator is

$$U_{\mathrm{F}} = \mathscr{T}^{1}_{-T_2} \exp[-iv(\theta_1)] \tag{4.4}$$

with

$$v(\theta_1) = -\int_0^{T_2} dt'\, f(\omega_1 t' + \theta_1, \omega_2 t') \tag{4.5}$$

Remark. A similar problem was treated in ref. 24 in a somewhat different context, involving periodically kicked scalar Hamiltonians. Many of those results can be readily translated into the present quasiperiodic case.

We first formulate a sufficient condition for point spectrum:

Theorem 4.1. If $\alpha = \omega_1/\omega_2$ satisfies a *diophantine* condition with $\gamma > 0$ and $\sigma > 1$ such that for all $\mathbf{n} \in \mathbb{Z}^2$ $(n_1 \neq 0)$

$$|\alpha n_1 + n_2| > \frac{\gamma}{|n_1|^\sigma} \tag{4.6}$$

and if f has a Fourier representation

$$f(\theta_1, \theta_2) = \sum_{\mathbf{n} \in \mathbb{Z}^2} \tilde{f}(\mathbf{n})\, e^{i\mathbf{n}\cdot\boldsymbol{\theta}} \tag{4.7}$$

such that

$$\sum_{\mathbf{n} \in \mathbb{Z}^2} |\tilde{f}(\mathbf{n})|\ |n_1|^\sigma < \infty \tag{4.8}$$

then the spectrum of the Floquet operator is pure point.

Remark. The complement of the set of α's satisfying (4.6) has zero measure. The conditions for this lemma could be substituted by the single condition

$$\sum_{\mathbf{n} \in \mathbb{Z}^2} |\tilde{f}(\mathbf{n})|/|\mathbf{n} \cdot \boldsymbol{\omega}| < \infty$$

Proof. Corollary 3.3 means in the scalar case that the eigenfunctions of U_F have constant absolute value. We can therefore represent them as

$$\phi(\theta_1) = e^{i\varphi(\theta_1)} \tag{4.9}$$

The eigenvalue equation can then be expressed as

$$v(\theta_1) + \varphi(\theta_1) = \mu + \varphi(\theta_1 + 2\pi\alpha) \pmod{2\pi} \tag{4.10}$$

with $\mu = -\lambda T_2$. The condition that $\phi \in L_2(S^1, d\theta_1)$ is equivalent to the requirement that φ is a measurable function. We will try to find eigenfunctions representing them by Fourier series:

$$\varphi(\theta_1) = \sum_{n_1 \in \mathbb{Z}} \tilde{\varphi}(n_1)\, e^{in_1\theta_1}, \qquad v(\theta_1) = \sum_{n_1 \in \mathbb{Z}} \tilde{v}(n_1)\, e^{in_1\theta_1} \tag{4.11}$$

Inserting into (4.10) with mod 2π omitted, we get

$$\tilde{v}(n_1) + \tilde{\varphi}(n_1) = \mu + \tilde{\varphi}(n_1)\, e^{2\pi i n_1 \alpha} \tag{4.12}$$

The equation for $n_1 = 0$ determines the eigenvalue: $\mu = \tilde{v}(0)$. We set $\tilde{\varphi}(0) = 0$, which is only a trivial phase. The other n_1 yield

$$\tilde{\varphi}(n_1) = \frac{\tilde{v}(n_1)}{e^{2\pi i n_1 \alpha} - 1}, \qquad n_1 \neq 0 \tag{4.13}$$

To show the convergence of the Fourier series, we first write

$$v(\theta_1) = \sum_{\mathbf{n}} \tilde{f}_{\mathbf{n}} e^{in_1\theta_1} \int_0^{T_2} dt'\, e^{i\mathbf{n}\cdot\boldsymbol{\omega} t'} \equiv \sum_{\mathbf{n}} \tilde{f}_{\mathbf{n}} \Gamma_{\mathbf{n}} e^{in_1\theta_1} \tag{4.14}$$

and therefore

$$\tilde{v}(n_1) = \sum_{n_2} \tilde{f}_{\mathbf{n}} \Gamma_{\mathbf{n}} \tag{4.15}$$

Since $|\Gamma_{\mathbf{n}}| \leqslant |T_2|$, we have

$$|\tilde{v}(n_1)| \leqslant |T_2| \sum_{n_2} |\tilde{f}_{\mathbf{n}}| \tag{4.16}$$

Using the inequality $|\sin \vartheta| \geqslant |\vartheta|\, 2/\pi$ for $\vartheta \in [-\pi/2, \pi/2]$, the denominator in (4.13) can be estimated by

$$\begin{aligned} |e^{2\pi i\alpha n_1} - 1| &\geqslant |\sin(2\pi\alpha n_1)| \\ &\geqslant \begin{cases} 2(\alpha 2n_1)_{\mathrm{mod}\ 1} & \text{if} \quad (\alpha 2n_1)_{\mathrm{mod}\ 1} \in [0, 1/2] \\ 2\,|(\alpha 2n_1)_{\mathrm{mod}\ 1} - 1| & \text{if} \quad (\alpha 2n_1)_{\mathrm{mod}\ 1} \in [1/2, 1] \end{cases} \end{aligned} \tag{4.17}$$

which combined with the Diophantine condition (4.6) gives

$$|\tilde{\varphi}(n_1)| < c_1\, |n_1|^\sigma \sum_{n_2} |\tilde{f}_{\mathbf{n}}| \tag{4.18}$$

Therefore

$$\sum_{n_1} |\tilde{\varphi}(n_1)| < c_1 \sum_{\mathbf{n}} |\tilde{f}_{\mathbf{n}}|\, |n_1|^\sigma < \infty \tag{4.19}$$

and thus the Fourier series converges.

Remarks. 1. It is clear from (4.13) and (4.15) that if $f(\boldsymbol{\theta})$ is a trigonometric polynomial, the spectrum is pure point for any irrational α.

2. From (4.13) and

$$|e^{2\pi i\alpha n_1} - 1| \leqslant 2\pi(\alpha n_1)_{\mathrm{mod}\ 1}$$

we can deduce an inequality in the opposite direction:

$$|\tilde{\varphi}(n_1)| > \frac{|\tilde{v}(n_1)|}{2\pi(\alpha n_1)_{\mathrm{mod}\ 1}} \tag{4.20}$$

By chosing an α that is well approximable by rationals (i.e., such that the terms in its continuous fraction representation grow fast enough), there always is an infinite subsequence of n_1 such that for v arbitrarily smooth but not a trigonometric polynomial

$$(\alpha n_1)_{\mathrm{mod}\ 1} < \frac{|\tilde{v}(n_1)|}{2\pi} \tag{4.21}$$

Then the Fourier series is not convergent. We cannot, however, conclude that in that case there are no eigenfunctions, since the only requirement for $\varphi(\theta_1)$ is that it is a measurable function, which does not entail the existence of a Fourier representation. Furthermore, Eq. (4.10) needs to be satisfied only modulo 2π, which again opens the possibility of solutions that do not have a Fourier representation. Therefore to prove that the spectrum is continuous one needs other methods.

The following result was proven in ref. 24.

Theorem 4.2. If there are γ_+, γ_- such that

$$\limsup_{|n_1| \to \infty} |\tilde{v}(n_1)|^{1/|n_1|} = e^{-\gamma_+} > 0, \qquad \liminf_{|n_1| \to \infty} |\tilde{v}(n_1)|^{1/|n_1|} = e^{-\gamma_-} > 0 \tag{4.22}$$

with $\gamma_+ \leqslant \gamma_- < 2\gamma_+$, and

$$\limsup_{|n_1| \to \infty} \frac{1}{|n_1|} \ln |\sin(\pi\alpha n_1)|^{-1} > 8\gamma_- \tag{4.23}$$

then the Floquet operator has purely continuous spectrum. Furthermore, if

$$\int_0^{2\pi} d\theta_1 \left| \frac{dv(\theta_1)}{d\theta_1} \right| < 1 \tag{4.24}$$

then there is no absolutely continuous spectrum. Thus, when the two sets of conditions are satisfied the spectrum is purely singular continuous.

Here we will prove a similar statement, but under different conditions:

Theorem 4.3. If

$$f(\boldsymbol{\theta}) \in C^1(\mathbb{T}^2) \tag{4.25}$$

and

$$\lim_{|\mathbf{n}| \to \infty} \frac{|\tilde{f}(\mathbf{n})|}{|\mathbf{n} \cdot \boldsymbol{\omega}|} = \infty \tag{4.26}$$

for some subsequence $\mathbf{n} = \mathbf{n}_k$, $k \in \mathbb{N}$, and there is an $\varepsilon_0 > 0$ such that for every $\mathbf{n} = \mathbf{n}_k$ from this subsequence either

$$|\tilde{f}(\mathbf{n})| \geqslant (1 + \varepsilon_0) \sum_{l=2}^{\infty} |\tilde{f}(l\mathbf{n})| \tag{4.27a}$$

or

$$|\tilde{f}(\mathbf{n})| \geqslant \varepsilon_0 \sum_{l=2}^{\infty} l\,|\tilde{f}(l\mathbf{n})| \tag{4.27b}$$

then the Floquet operator has purely continuous spectrum.

The proof is given in the Appendix.

Remark. In Theorems 4.2 and 4.3 one finds continuous spectrum for smooth (or analytic) Hamiltonians under some Liouville-type condition for α. If one allows for Hamiltonians that are not continuous, it is easy to find examples that have continuous spectrum. For instance, if $v(\theta_1) = (\theta_1)_{\mathrm{mod}\,2\pi}$, the spectrum is absolutely continuous.[24] The following example was formulated by Kesten.[30]

Theorem 4.4. Define

$$v(\theta_1) = \begin{cases} 1, & 0 \leqslant \theta_1 \leqslant a \\ 0, & a < \theta_1 < 2\pi \end{cases} \tag{4.28}$$

and continued periodically. If $a/(2\pi)$ is rational, then the spectrum of U_F is purely continuous for all irrational α.

5. CONTINUOUS SPECTRUM IN TWO-LEVEL SYSTEMS

The general Hamiltonian acting on $\mathscr{H} = \mathbb{C}^2$ is of the form

$$H = h_0(t)\,\mathbb{1} + \sum_{j=1}^{3} h_j(t)\,\sigma_j \tag{5.1}$$

where σ_j are the Pauli matrices, and $h_j(t)$ are real quasiperiodic functions, i.e., $h_j(t) = \bar{h}_j(\omega_1 t + \theta_1, \omega_2 t + \theta_2)$, where $\bar{h}_j(\theta_1, \theta_2)$ are continuous and 2π-periodic in the two arguments $\theta_1, \theta_2 \in S^1$. Without loss of generality we can take $h_0 = 0$, since its effect is only a global time-dependent phase that can be studied with the methods of the previous section. The Hamiltonian is then a Hermitian 2×2 matrix with trace zero, and thus the propagator $U(t, s; \boldsymbol{\theta})$ is unitary with determinant one [i.e., $\in SU(2)$].

In this section we will discuss an example proposed by Rychlik[27] that has continuous spectrum for any frequency ratio α. We start with some general remarks. The Floquet operator is $U_F = \mathscr{T}_{-T_2} u_1$ with the monodromy operator $u_1 \equiv U(T_2, 0; \theta_1, 0) \in SU(2)$, i.e., it can be represented in general as

$$u_1(\theta_1) = \begin{pmatrix} a & b^* \\ -b & a^* \end{pmatrix}; \qquad |a|^2 + |b|^2 = 1 \tag{5.2}$$

Lemma 5.1. For any choice of C^l-functions a, b from S^1 to $\mathbb{C}$, $l=0, 1, \ldots, \infty$, with $|a|^2+|b|^2=1$ there is some quasiperiodic H of the form (5.1) such that (5.2) is the corresponding monodromy operator. In fact, there is large family of them, and they can be chosen to be as smooth as a and b.

Proof. We first remark that if the propagator $U(t, 0; \theta_1, 0)$ is given for all $\theta_1 \in S^1$ and all $t \in [0, T_2]$, then it is completely determined for all t: According to (3.19), if $u_1(\theta_1)$ is given for all $\theta_1 \in S^1$, then $U(kT_2, 0; \theta_1, 0) \equiv u_k(\theta_1)$ is determined for all times that are integer multiples of T_2. We can decompose $t = kT_2 + \delta t$, $k \in \mathbb{Z}$, $|\delta t| < T_2$. Then

$$\begin{aligned} U(t, 0; \theta_1, 0) &= U(kT_2 + \delta t, 0; \theta_1, 0) \\ &= U(kT_2 + \delta t, kT_2; \theta_1, 0)\, U(kT_2, 0; \theta_1, 0) \\ &= U(\delta t, 0; \theta_1 + \omega_1 kT_2, 0)\, u_k(\theta_1) \end{aligned} \tag{5.3}$$

For the construction of the quasiperiodic Hamiltonian we first construct a propagator for $t \in [0, T_2]$ and then extend it to $t > T_2$ using (5.3). In order to obtain a smooth Hamiltonian we have to make sure that the different pieces match properly at $t = kT_2$.

The set $\{u_1(\theta_1), \theta_1 \in S^1\} \in SU(2)$ is topologically a circle, since it is the image of S^1 by a continuous map. We can construct a function $v(t; \theta_1)$, $t \in [0, T_2]$, that interpolates smoothly between the identity $\mathbb{1}$ at $t=0$ and $u_1(\theta_1)$ at $t = T_2$, such that for any fixed t, $v(t; \theta_1) \in SU(2)$. This is always possible because $SU(2)$ is simply connected and thus any circle $u_1(\theta_1)$ can be continuously deformed into a point ($\mathbb{1}$). With this v we define for $t = kT_2 + \delta t$

$$U(t, 0; \theta_1, 0) = v(\delta t; \theta_1 + \omega_1 kT_2)\, u_k(\theta_1) \tag{5.4}$$

Then we construct a first Hamiltonian $h(t)$ by

$$\begin{aligned} h(t) &= \left(i \frac{\partial}{\partial t} U(t, 0; \theta_1, 0) \right) (U(t, 0; \theta_1, 0))^{-1} \\ &= \left(i \frac{\partial}{\partial t} v(\delta t; \theta_1 + \omega_1 kT_2) \right) (v(\delta t; \theta_1 + \omega_1 kT_2))^{-1} \\ &\equiv V(\delta t; \theta_1 + \omega_1 kT_2) \end{aligned} \tag{5.5}$$

We require that the derivatives of $v(t; \theta_1)$ with respect to t are zero at $t=0$ and at $t = T_2$, which guarantees that $h(t)$ is smooth at the points $t = kT_2$.

We proceed as follows in order to verify that $h(t)$ is quasiperiodic and to construct from it a more general Hamiltonian depending on two parameters θ_1, θ_2: Taking into account that

$$\delta t = (t)_{\mathrm{mod}\, T_2} = \frac{1}{\omega_2}(\omega_2 t)_{\mathrm{mod}\, 2\pi}, \qquad kT_2 = t - \frac{1}{\omega_2}(\omega_2 t)_{\mathrm{mod}\, 2\pi} \tag{5.6}$$

and that the function V is 2π-periodic in the second argument, we can write

$$\begin{aligned} h(t) &= V\left(\frac{1}{\omega_2}(\omega_2 t)_{\mathrm{mod}\, 2\pi}; \left(\theta_1 + \omega_1 t - \frac{\omega_1}{\omega_2}(\omega_2 t)_{\mathrm{mod}\, 2\pi}\right)_{\mathrm{mod}\, 2\pi}\right) \\ &= V\left(\frac{1}{\omega_2}(\omega_2 t)_{\mathrm{mod}\, 2\pi}; (\theta_1 + \omega_1 t)_{\mathrm{mod}\, 2\pi} - \frac{\omega_1}{\omega_2}(\omega_2 t)_{\mathrm{mod}\, 2\pi}\right) \end{aligned} \tag{5.7}$$

Defining the 2π-periodic function in two arguments

$$A(\vartheta_1, \vartheta_2) = V\left(\frac{1}{\omega_2}(\vartheta_2)_{\mathrm{mod}\, 2\pi}; (\vartheta_1)_{\mathrm{mod}\, 2\pi} - \frac{\omega_1}{\omega_2}(\vartheta_2)_{\mathrm{mod}\, 2\pi}\right) \tag{5.8}$$

we can define a quasiperiodic Hamiltonian

$$H = A(\omega_1 t + \theta_1, \omega_2 t + \theta_2) \tag{5.9}$$

that by construction has u_1 as monodromy matrix.

Remark. The preceding construction cannot be done in the scalar case because there instead of $SU(2)$ we have $U(1)$, which is not simply connected.

We consider the following example[27]:

Proposition 5.2. Let

$$u_1(\theta_1) = \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{-i\theta_1} \end{pmatrix} \tag{5.10}$$

The corresponding Floquet operator $U_F = \mathscr{T}^{1}_{-T_2} u_1$ has absolutely continuous spectrum for any irrational α. More specifically, it is a Lebesgue spectrum.[25]

Proof. First we remark that there are no eigenvalues: Since the matrix u_1 is diagonal, we can write any candidate for eigenvector of U_F as $(y(\theta_1), 0)$ or $(0, z(\theta_1))$. For the first type, the eigenvalue equation is equivalent to

$$e^{i\theta_1} y(\theta_1) = e^{-i\mu} y(\theta_1 + 2\pi\alpha) \tag{5.11}$$

If there were an eigenfunction $y \in L_2(S^1, d\theta_1)$, it would necessarily have a converging Fourier series, with coefficients $\tilde{y}_n$. Insertion into (5.11) yields

$$\tilde{y}_{n+1} = e^{-i\mu} e^{-i\alpha n} \tilde{y}_n \tag{5.12}$$

which implies $|y_{n+1}| = |y_n|$ and therefore the Fourier series would not be convergent. As a consequence there are no eigenvalues.

Further, we consider the following basis of $\mathscr{K}_1 = \mathbb{C}^2 \otimes L_2(S^1, d\theta_1)$:

$$\{e_k, d_k\} \equiv \left\{ \begin{pmatrix} e^{i(k\theta_1 + \beta_k)} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ e^{-i(k\theta_1 + \beta_k)} \end{pmatrix} \right\} \tag{5.13}$$

where $\beta_k = 2\pi\alpha k(k+1)/2$. Then $U_F e_k = e_{k+1}$ and $U_F d_k = d_{k+1}$. Therefore,

$$\begin{aligned} \langle e_k, U_F^j e_k \rangle = \langle e_k, e_{k+j} \rangle = \delta_{j,0} = \int_0^{2\pi} \frac{d\theta_1}{2\pi} e^{-ij\theta_1} \\ = \langle d_k, U_F^j d_k \rangle \end{aligned} \tag{5.14}$$

i.e., the spectral measure associated the basis vectors is equal to the Lebesgue measure.

Remarks. 1. We say that the continuous spectrum of this example has a topological origin because of the following property: Corollary 3.3 implies that $y(\theta_1)$ and $z(\theta_1)$ have constant absolute value for almost all θ_1, and therefore we can write, e.g., $y(\theta_1) = \exp(i\varphi(\theta_1))$. The index of a function from S^1 to S^1 is defined as the number of times that the image wraps around the circle when θ_1 goes from 0 to 2π. In Eq. (5.11) the index of the left-hand side is larger by one than of the right-hand side, and therefore the equation cannot have a solution.

2. By the same mechanism, any monodromy operator of the form

$$u_1(\theta_1) = \begin{pmatrix} e^{iL\theta_1} & 0 \\ 0 & e^{-iL\theta_1} \end{pmatrix}, \qquad 0 \neq L \in \mathbb{Z} \tag{5.15}$$

leads to a Floquet operator with Lebesgue spectrum. By taking linear combinations and a suitable limit, we conclude that a monodromy operator of the form

$$u_1(\theta_1) = \begin{pmatrix} f(e^{iL\theta_1}) & 0 \\ 0 & f^*(e^{iL\theta_1}) \end{pmatrix}, \qquad L \neq 0 \in \mathbb{Z} \tag{5.16}$$

where f is an analytic function, leads to a Floquet operator with an absolutely continuous spectrum.

By Lemma 5.1 there is a smooth quasiperiodic Hamiltonian that has (5.10) as its monodromy operator. We will construct one example explicitly by the following procedure:

In order to construct the homotopy linking the identity with $u_1(\theta_1)$, we notice that (5.2) implies that $SU(2)$ is isomorphic to the three-sphere S^3. Embedded in $\mathbb{R}^4$ it can be parametrized by $\{\mathbf{x} \in \mathbb{R}^4;\ \sum_{i=1,4} |x_i|^2 = 1\}$, or by three angles

$$\begin{aligned} x_1 &= \sin\zeta \sin\varphi \sin\vartheta, \qquad & x_3 &= \sin\zeta \cos\vartheta \\ x_2 &= \sin\zeta \cos\varphi \sin\vartheta, \qquad & x_4 &= \cos\zeta \end{aligned} \tag{5.17}$$

We identify the real and imaginary parts of a and b with

$$\Re a = x_1, \qquad \Im a = x_2, \qquad \Re b = x_4, \qquad \Im b = x_3 \tag{5.18}$$

With this parametrization the identity of $SU(2)$ corresponds to $(1, 0, 0, 0)$, i.e., $(\zeta = \pi/2, \varphi = 0, \vartheta = \pi/2)$, and $u_1(\theta_1)$ to $(\cos\theta_1, \sin\theta_1, 0, 0)$, i.e., $(\zeta = \pi/2, \varphi = \theta_1, \vartheta = \pi/2)$.

The homotopy is given by a set of smooth functions $x_i(t)$, $i = 1,..., 4$, $t \in [0, T_2]$. We will choose the special subset of homotopies in which $x_4(t) = 0$ is kept constant. This allows us to work with the two-sphere S^2 embedded in $\mathbb{R}^3$. The identity is the point on the x_1 axis with $x_1 = 1$, and U_F is the circle of the equator on the x_1, x_2 plane. We deform this circle into the point of the identity by a combination of two transformations

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = R_2(t)\, T_3(t) \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \\ 0 \end{pmatrix} \tag{5.19}$$

The first one pulls the circle along the x_3 axis:

$$T_3(t) \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sin\delta(t)\cos\theta_1 \\ \sin\delta(t)\sin\theta_1 \\ \cos\delta(t) \end{pmatrix} \tag{5.20}$$

where $\delta(t)$ is a monotonically increasing function with $\delta(0) = 0$ and $\delta(T_2) = \pi/2$. The second transformation is a rotation around the x_2 axis

$$R_2(t) = \begin{pmatrix} \cos[\pi/2 - \delta(t)] & 0 & \sin[\pi/2 - \delta(t)] \\ 0 & 1 & 0 \\ -\sin[\pi/2 - \delta(t)] & 0 & \cos[\pi/2 - \delta(t)] \end{pmatrix} \tag{5.21}$$

Combining them, we obtain

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} \cos^2 \delta(t) + \sin^2 \delta(t) \cos \theta_1 \\ \sin \delta(t) \sin \theta_1 \\ \sin \delta(t) \cos \delta(t)(1 - \cos \theta_1) \end{pmatrix} \tag{5.22}$$

This homotopy is illustrated in Fig. 2.

Applying (5.18) and (5.7), we write $a(t; \theta_1) = x_1 + ix_2$, $b(t; \theta_1) = ix_3$, and the Hamiltonian as

$$h(t) = \begin{pmatrix} c(t, \theta_1) & g(t, \theta_1) \\ g^*(t, \theta_1) & -c(t, \theta_1) \end{pmatrix} \tag{5.23}$$

with

$$c(t, \theta_1) = i\left(a^* \frac{da}{dt} + b \frac{db^*}{dt}\right) \in \mathbb{R},$$

$$g(t, \theta_1) = i\left(-b^* \frac{da}{dt} + a \frac{db^*}{dt}\right) \in \mathbb{C} \tag{5.24}$$

Inserting (5.22), one obtains

$$c(t, \theta_1) = [-\cos \delta(1 + \sin^2 \delta) \sin \theta_1 + \cos \delta \sin^2 \delta \cos \theta_1 \sin \theta_1] \frac{d\delta}{dt}$$

$$\begin{aligned} g(t, \theta_1) = \{&\sin^3 \delta \sin \theta_1 (1 - \cos \theta_1) \\ &+ i[\cos^2 \delta \sin^2 \theta_1 - \cos \theta_1 (1 - \cos \theta_1)]\} \frac{d\delta}{dt} \end{aligned} \tag{5.25}$$

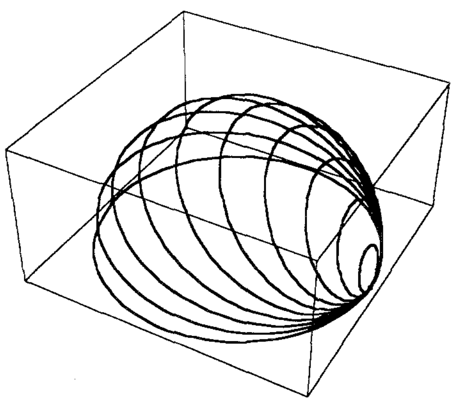

Fig. 2. Interpolation between the identity and $u_1(\theta_1)$.

This expression shows that if we choose $\delta(t)$ in C^k and such that its derivatives up to order k are zero at $t=0$ and at $t=T_2$ we obtain a quasiperiodic Hamiltonian that is C^{k-1}. The simplest example is

$$\delta(t)=\pi(t/T_2)^2\,(3/2-t/T_2) \tag{5.26}$$

which leads to a continuous Hamiltonian.

Remark. We do not know if it is possible to obtain continuous spectrum for any irrational α in the case of analytic Hamiltonians.

6. STABILITY OF THE POINT SPECTRUM

As a consequence of the results of Section 2, when the spectrum of the quasienergy operator is pure point it has the structure

$$\lambda_{\mathbf{n},m}=\bar{\lambda}_m+n_1\omega_1+n_2\omega_2, \qquad m=1,..., N, \quad \mathbf{n}\in\mathbb{Z}^2 \tag{6.1}$$

i.e., it is a dense subset of $\mathbb{R}$. Therefore one cannot apply the usual perturbation theory,[31] since already in the first-order terms there appear sums whose convergence is not clear due to the presence of small denominators. Under suitable Diophantine conditions on the frequencies one can control the convergence of the perturbation series by using a KAM-type technique introduced by Bellissard.[24]

Theorem 6.1. Let

$$K=-i\omega_1\frac{\partial}{\partial\theta_1}-i\omega_2\frac{\partial}{\partial\theta_2}+H_0+\varepsilon V(\theta_1,\theta_2)\equiv K_0+\varepsilon V(\theta_1,\theta_2) \tag{6.2}$$

where H_0 and $V(\theta_1,\theta_2)$ are Hermitian 2×2 matrices, H_0 being constant and $V(\theta_1,\theta_2)$ such that each component is an analytic function in the strip $\{\boldsymbol{\theta}\,|\,\mathrm{Im}\,\theta_j<r_0\}$. Assume, e.g., that $\tilde{\alpha}=\omega_2/\omega_1>1$, and that $(2\beta/\omega_1)_{\mathrm{mod}\,1}>0$, where $2\beta>0$ is the difference between the two eigenvalues of H_0. Then, for any given $\eta>0$ and fixed ω_1 there is a set of $\tilde{\alpha}$'s $S_\eta\subset(1,\infty)$ of Lebesgue measure $|S_\eta|<\eta$ and a value $\varepsilon_c(\eta)$ such that if $\tilde{\alpha}\in(1,\infty)\backslash S_\eta$ and $\varepsilon<\varepsilon_c$, the spectrum of K is pure point.

Proof. The proof follows the scheme that was set up by Bellissard for a similar theorem on the smoothly kicked rotator[24] and on the ac Stark effect,[32] combined with some improvements introduced by Combescure[33] in her treatment of perturbations of the harmonic oscillator. Without loss of generality we set

$$H_0 = \beta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6.3}$$

For the proof we construct a unitary transformation $R(\bar{\alpha}, \varepsilon)$ such that

$$RKR^{-1} = K^{(0)} + \begin{pmatrix} g_+(\bar{\alpha}, \varepsilon) & 0 \\ 0 & g_-(\bar{\alpha}, \varepsilon) \end{pmatrix} \tag{6.4}$$

where $g_\pm$ are independent of $\boldsymbol{\theta} \equiv (\theta_1, \theta_2)$. Stated equivalently, K is transformed to an operator that is diagonal in the basis of eigenfunctions of the unperturbed $K^{(0)}$, which are of the form

$$\psi_{\mathbf{n},m} = \begin{cases} e^{i\mathbf{n}\cdot\boldsymbol{\theta}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & \text{if } m = +1 \\ e^{i\mathbf{n}\cdot\boldsymbol{\theta}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, & \text{if } m = -1 \end{cases} \qquad \mathbf{n} \equiv (n_1, n_2) \in \mathbb{Z}^2; \quad m \in \{+1, -1\} \tag{6.5}$$

corresponding to eigenvalues $\boldsymbol{\omega} \cdot \mathbf{n} \pm \beta$, where $\boldsymbol{\omega} = (\omega_1, \omega_2)$. The transformation R is constructed by iteration $R = \ldots R_k \ldots R_2 R_1$; at each step the order of the $\boldsymbol{\theta}$-dependent perturbation is reduced from order ε^j to order ε^{2j}.

Remark. The comparison of orders of ε is used only in informal arguments, in order to guess the form of the transformation. Once this form is fixed, the proof proceeds by estimates for finite ε, and one does not need to worry about the classification of terms in orders of ε.

The kth iteration step is defined as follows: We start with an operator of the form

$$K_k = D_k + V_k; \qquad D_k \equiv K^{(0)} + g_k \tag{6.6}$$

where g_k and D_k are diagonal in the basis $\{\psi_{\mathbf{n},m}\}$ and V_k is Hermitian. g_k has been generated in the previous iterations and can depend explicitly on $\bar{\alpha}$, but it has no dependence on $\boldsymbol{\theta}$. We represent the transformation as $R_{k+1} = e^{W_{k+1}}$, with $W_{k+1}^\dagger = -W_{k+1}$. Expanding the exponentials and regrouping the terms that we expect to be of the same order as V_k (i.e., making the hypothesis that will be justified *a posteriori* that W_{k+1} is of the same order as V_k), we obtain

$$\begin{aligned} K_{k+1} &\equiv e^{W_{k+1}} K_k e^{-W_{k+1}} \\ &= K_k + [W_{k+1}, K_k] + \frac{1}{2!}[W_{k+1}, [W_{k+1}, K_k]] \\ &\quad + \frac{1}{3!}[W_{k+1}, [W_{k+1}, [W_{k+1}, K_k]]] + \cdots \\ &= D_k + V_k + [W_{k+1}, D_k] + V_{k+1} \end{aligned} \tag{6.7}$$

where we have selected

$$\begin{aligned} V_{k+1} = &\left[W_{k+1}, \left\{ [W_{k+1}, D_k] \frac{1}{2!} + V_k \right\} \right] \\ &+ \left[W_{k+1}, \left[W_{k+1}, \left\{ [W_{k+1}, D_k] \frac{1}{3!} + V_k \frac{1}{2!} \right\} \right] \right] + \cdots \end{aligned} \tag{6.8}$$

We will work with the matrix representation corresponding to the basis $\{\psi_{\mathbf{n},m}\}$, using the notation

$$W_{k+1}(m, m', \mathbf{n} - \mathbf{n}') \equiv \langle \psi_{\mathbf{n},m}, W_{k+1} \psi_{\mathbf{n}',m'} \rangle \tag{6.9}$$

The matrix elements depend only on the difference $\mathbf{n} - \mathbf{n}'$ because of the special form (6.5) of the eigenfunctions $\psi_{\mathbf{n},m}$. For the diagonal operators D we use the simplified notation

$$D(m, \mathbf{n}) \equiv \langle \psi_{\mathbf{n},m}, D\psi_{\mathbf{n},m} \rangle \tag{6.10}$$

and

$$g_k(m) \equiv \langle \psi_{\mathbf{n},m}, g_k \psi_{\mathbf{n},m} \rangle \tag{6.11}$$

which is independent of $\mathbf{n}$ since g_k has no dependence on $\boldsymbol{\theta}$.

Motivated by the decomposition (6.7), we proceed as follows:

1. Determine W_{k+1} and a diagonal δg such that

$$e^{W_{k+1}} K_k e^{-W_{k+1}} = D_k + \delta g + V_{k+1} \tag{6.12}$$

This will be satisfied if W_{k+1} and δg satisfy the equation

$$[W_{k+1}, D_k] + V_k = \delta g \tag{6.13}$$

which one can solve explicitly: The diagonal terms of (6.13) yield

$$\delta g(m) = V_k(m, m, \mathbf{0}) \tag{6.14}$$

and one can choose $W_{k+1}(m, m, \mathbf{0}) = 0$. The off-diagonal terms yield

$$W_{k+1}(m, m', \mathbf{n}-\mathbf{n}') = \frac{V_k(m, m', \mathbf{n})}{D_k(m, \mathbf{n}) - D_k(m', \mathbf{n}')} \tag{6.15}$$

The denominator

$$D_k(m, \mathbf{n}) - D_k(m', \mathbf{n}') = \boldsymbol{\omega} \cdot (\mathbf{n}-\mathbf{n}') + (m-m')\,\beta + g_k(m) - g_k(m') \tag{6.16}$$

can be zero or arbitrarily close to zero for infinitely many indices m, m', $\mathbf{n}-\mathbf{n}'$. To guarantee the convergence of the series (6.15) and to obtain suitable estimates, we fix ω_1 and restrict the values of $\bar{\alpha}$ to the ones belonging to the set characterized by the following Diophantine condition:

$$\Omega_{k+1}(\gamma_{k+1}) = \Bigg\{ \bar{\alpha} \in \Omega_k(\gamma_k);\ \text{such that } \forall \mathbf{n} \in \mathbb{Z}^2, \text{ and } m, m' \in \{+1, -1\},$$

$$|\boldsymbol{\omega} \cdot \mathbf{n} + (m-m')\,\beta + g_k(m) - g_k(m')| \geqslant \frac{\gamma_{k+1}}{(1+|\mathbf{n}|)^\sigma} \Bigg\} \tag{6.17}$$

where $\mathbf{n}$ and $m - m'$ are not simultaneously zero, $\Omega_0 = (1, \infty)$, $\sigma > 2$ is fixed, and γ_{k+1} is a constant that we will choose for each step. As we will show, the Lebesgue measure $\mathscr{L}$ of the complement of this set is small. Since the g_k in the denominator depend on $\bar{\alpha}$, in order to get estimates of Diophantine type, we need to keep control on the size of g_k as well as on its variation as a function of $\bar{\alpha}$. With this motivation, we define the following norm on operators A that depend parametrically on $\bar{\alpha} \in \Omega$,[33]

$$\|A\|_{r,\Omega} = \sum_{\mathbf{n}, \Delta m} e^{r|\mathbf{n}|} \sup_{\bar{\alpha}, \bar{\alpha}' \in \Omega} \sup_{m} \Bigg(|A(m, m+\Delta m, \mathbf{n}; \bar{\alpha})|$$

$$+ \frac{|A(m, m+\Delta m, \mathbf{n}; \bar{\alpha}) - A(m, m+\Delta m, \mathbf{n}; \bar{\alpha}')|}{|\bar{\alpha} - \bar{\alpha}'|} \Bigg) \tag{6.18}$$

We remark that the use of a finite difference in (6.18) instead of a derivative with respect to $\bar{\alpha}$ is necessary, since, as the iteration proceeds, g_k is only defined on subsets of $\mathbb{R}$ that have many gaps.

We will use the shorter notation $\|\cdot\|_k \equiv \|\cdot\|_{r_k, \Omega_k}$.

Lemma 6.1. We denote $\zeta \equiv \inf_{j \in \mathbb{Z}} |2\beta/\omega_1 - j|$, i.e., the distance from $2\beta/\omega_1$ to the closest integer. For $\sigma > 2$, if

$$\|g_k\|_k < \tfrac{1}{4}\omega_1 \zeta \tag{6.19}$$

and

$$\gamma_{k+1} < \frac{\omega_1}{2} \zeta \tag{6.20}$$

then the Lebesgue measure of $\Omega_k \backslash \Omega_{k+1}$ is bounded by

$$\mathscr{L}(\Omega_k \backslash \Omega_{k+1}) < c_1(\sigma)\, \gamma_{k+1} \tag{6.21}$$

where $c_1(\sigma)$ is a constant.

2. Denoting $\varepsilon_k = \| V_k \|_k$, we obtain the following estimates for the solution of Eq. (6.13):

Lemma 6.2. For $0 < r_{k+1} < r_k$ and $\gamma_k < 1$,

$$\| W_{k+1} \|_{k+1} \leqslant A_k \varepsilon_k \tag{6.22}$$

with

$$A_k = \frac{c_0(\sigma)}{(\gamma_{k+1})^2\, (\rho_{k+1})^{2\sigma+1}}, \qquad \rho_{k+1} \equiv r_k - r_{k+1} \tag{6.23}$$

and

$$\| \delta g_{k+1} \|_{k+1} < \varepsilon_k \tag{6.24}$$

3. We need to show that the composition of transformations $S_k \equiv R_k \ldots R_2 R_1$ converges. This will be the case under the following condition:

Lemma 6.3. If

$$\sum_{k=1}^{\infty} \| W_k \|_k < a < \infty \tag{6.25}$$

then S_k converges to a unitary transformation: $\lim_{k \to \infty} S_k = R$.

4. The remainder term of (6.12) is estimated by the following result.

Lemma 6.4. We have

$$\varepsilon_{k+1} \leqslant 2\, \| W_{k+1} \|_{k+1}\, \varepsilon_k \exp(2\, \| W_{k+1} \|_{k+1}) \tag{6.26}$$

Combining the estimates (6.22) and (6.26), we obtain the recursive inequality

$$\varepsilon_{k+1} \leqslant A_k \varepsilon_k^2 \exp(A_k \varepsilon_k) \tag{6.27}$$

5. With the choices

$$\gamma_k = \gamma_\infty / 2^{k+1}, \qquad \gamma_\infty < \omega_1 \zeta / 4; \qquad \rho_k = \rho_\infty / 2^{k+1} \tag{6.28}$$

where $\zeta = \inf_{j \in \mathbb{Z}} |2\beta/\omega_1 - j|$, we can write

$$A_k = B 2^{\nu k}; \qquad \text{where} \quad B = \frac{2^{\nu+1} c_0(\sigma)}{\gamma_\infty^2 \rho_\infty^{2\sigma+1}}; \quad \nu = 2\sigma + 3 \tag{6.29}$$

The inequalities (6.27) imply then the following:

Lemma 6.5. If $\varepsilon_0 < \min\{\omega_1 \zeta (2^\nu - 1)/2^{\nu+2}, \quad 1/(B 2^\nu e)\}$, then $\lim_{k \to \infty} \varepsilon_k = 0$. More precisely, there is a positive $\mu < 1$ such that:

(i) $\varepsilon_k \leqslant c_4 2^{-\nu k} \mu^{2^k}$, $c_4 = \text{const}$

(ii) $\| W_k \|_{k+1} < c_5 \mu^{2^k}$, $c_5 = \text{const}$

(iii) $\sum_{k=0}^\infty \varepsilon_k < \omega_1 \zeta / 4$

The statements (i) and (ii) guarantee that the iteration converges. Property (ii) implies that $\sum_k \| W_k \|_k$ is finite, which is the condition needed in Lemma 6.3 for the convergence of the unitary transformations.

Property (iii) implies that $g_\pm$ is finite [and in particular at each step $\| g_k \|_k = \sum_{k'=1}^k \| \delta g_{k'} \|_{k'} < \sum_{k'=1}^k \varepsilon_{k'} < \omega_1 \zeta / 4$, which is the condition (6.19) needed for the estimate in Lemma 6.1 of the measure of the sets $\Omega_k \backslash \Omega_{k+1}$].

6. The choice of the constants γ_k was made such that the total measure of the subset of $\bar{\alpha}$'s that we have to exclude can be estimated by

$$\mathscr{L}\left(\bigcup_k (1, \infty) \backslash \Omega_k \right) < c_1(\sigma) \sum_{k=0}^\infty \gamma_k = c_1(\sigma) \gamma_\infty \tag{6.30}$$

i.e., the η appearing in the statement of the theorem is identified with $\eta \equiv c_1(\sigma) \gamma_\infty$.

Proof of Lemma 6.1. We start by considering the set defined for $\mathbf{n}$ or $m - m'$ different from zero as

$$I_{\mathbf{n}}(\xi) = \{ \bar{\alpha} \in \Omega_k; \, |\boldsymbol{\omega} \cdot \mathbf{n} + (m - m') \beta + G(m, m', \bar{\alpha})| \leqslant \xi \} \tag{6.31}$$

where $G(m, m', \bar{\alpha}) \equiv g_k(m) - g_k(m')$. We show that if

$$\xi < \omega_1 \zeta / 2 \tag{6.32}$$

then

$$\mathscr{L}(I_{\mathbf{n}}(\xi)) \leqslant \frac{4}{\omega_1} \xi \tag{6.33}$$

The condition (6.19) implies that $|G| < \frac{1}{2}\omega_1\zeta$, which together with (6.32) implies that the set $I_{\mathbf{n}}(\xi)$ is empty unless $n_2 \neq 0$. In that case consider two $\bar{\alpha}, \bar{\alpha}' \in I_{\mathbf{n}}(\xi)$, i.e., both satisfy

$$|n_1 + \bar{\alpha}n_2 + (m - m')\,\beta/\omega_1 + G(m, m', \bar{\alpha})/\omega_1| \leqslant \xi/\omega_1 \tag{6.34}$$

Subtracting the two inequalities, we get

$$|\bar{\alpha} - \bar{\alpha}'| \left| n_2 + \frac{G(\bar{\alpha}) - G(\bar{\alpha}')}{(\bar{\alpha} - \bar{\alpha}')\,\omega_1} \right| < \frac{2\xi}{\omega_1} \tag{6.35}$$

The condition (6.19) implies

$$\frac{|G(\bar{\alpha}) - G(\bar{\alpha}')|}{|\bar{\alpha} - \bar{\alpha}'|} < \frac{\omega_1}{2} \tag{6.36}$$

and therefore

$$|\bar{\alpha} - \bar{\alpha}'| < \frac{4}{\omega_1}\,\xi \tag{6.37}$$

which gives the bound on the measure of $I_{\mathbf{n}}(\xi)$ that is independent of $\mathbf{n}$.

With this result we proceed by summing over all indices of the set $\Gamma(N) = \{\mathbf{n} \in \mathbb{Z}^2;\ |\mathbf{n}| = N\}$ which contains $4N$ elements:

$$\sum_{\mathbf{n} \in \Gamma(N)} \mathscr{L}(I_{\mathbf{n}}(\xi)) \leqslant \frac{16N}{\omega_1}\,\xi \tag{6.38}$$

We then choose $\xi = \gamma_{k+1}/(|\mathbf{n}| + 1)^\sigma$ and obtain that the measure of the set

$$\begin{aligned} I_{\Gamma(N)} = \{\bar{\alpha} \in \Omega_k;\ &|\mathbf{w} \cdot \mathbf{n} + (m - m')\,\beta + G(m, m', \bar{\alpha})| \\ &\leqslant \gamma_{k+1}/(|\mathbf{n}| + 1)^\sigma \text{ for some } \mathbf{n} \in \Gamma(N)\} \end{aligned} \tag{6.39}$$

is bounded by

$$\mathscr{L}(I_{\Gamma(N)}) < \frac{16N\gamma_{k+1}}{(|\mathbf{n}| + 1)^\sigma} < 16N^{1-\sigma}\gamma_{k+1} \tag{6.40}$$

and summing over N

$$\mathscr{L}(\Omega_k \backslash \Omega_{k+1}) = \sum_{N=1}^{\infty} \mathscr{L}(I_{\Gamma(N)}) < 16\gamma_{k+1} \sum_{N=1}^{\infty} N^{1-\sigma} \equiv c_1(\sigma)\,\gamma_{k+1} \tag{6.41}$$

where the last sum converges, since $\sigma > 2$.

Proof of Lemma 6.2. We will use the shortened notation $d = \boldsymbol{\omega}\cdot\mathbf{n} + (m-m')\beta + G(m,m;\bar{\alpha})$, $V = V_k(\mathbf{n},m,m';\bar{\alpha})$, and $W = W_{k+1}(\mathbf{n},m,m';\bar{\alpha})$; and d', V', W' and G' are the same quantities, but evaluated at $\bar{\alpha}'$.

To estimate the norm $\|W_{k+1}\|_{k+1}$ as defined by (6.15), we use (6.17): $1/|d| < (|\mathbf{n}|+1)^{\sigma}/\gamma_{k+1}$.

We start with

$$|W| = \frac{|V|}{|d|} < |V|\,\frac{(1+|\mathbf{n}|)^{\sigma}}{\gamma_{k+1}} < |V|\,\frac{(1+|\mathbf{n}|)^{2\sigma+1}}{\gamma_{k+1}^2} \tag{6.42}$$

where we have used $|\gamma_{k+1}| < 1$. Further, we decompose

$$\begin{aligned} |W-W'| &= \left|\frac{V}{d} - \frac{V'}{d'}\right| = \left|V\left(\frac{1}{d}-\frac{1}{d'}\right) + \frac{1}{d'}(V-V')\right| \\ &\leqslant |V|\left|\frac{d-d'}{dd'}\right| + \frac{1}{|d'|}|V-V'| \end{aligned} \tag{6.43}$$

For the first term we write

$$\begin{aligned} \left|\frac{d-d'}{dd'}\right| &= \left|\frac{(\bar{\alpha}-\bar{\alpha}')\,n_2 - (G-G')/\omega_1}{dd'}\right| \\ &< \left|\frac{(\bar{\alpha}-\bar{\alpha}')\,n_2 - (G-G')/\omega_1}{\gamma_{k+1}^2(|\mathbf{n}|+1)^{-2\sigma}}\right| \\ &< \frac{(|\mathbf{n}|+1)^{2\sigma+1}}{\gamma_{k+1}^2}\left(|\bar{\alpha}-\bar{\alpha}'| + \frac{|G-G'|}{\omega_1}\right) \\ &< \frac{(|\mathbf{n}|+1)^{2\sigma+1}}{\gamma_{k+1}^2}\,2\,|\bar{\alpha}-\bar{\alpha}'| \end{aligned} \tag{6.44}$$

For the last inequality we have used (6.36). Putting together (6.43) and (6.44), we obtain

$$\frac{|W-W'|}{|\bar{\alpha}-\bar{\alpha}'|} < \frac{(|\mathbf{n}|+1)^{2\sigma+1}}{\gamma_{k+1}^2}\left(|V| + \frac{|V-V'|}{|\bar{\alpha}-\bar{\alpha}'|}\right) \tag{6.45}$$

which together with (6.42) and the fact that for $0<\rho<r_0$, $2\sigma+1>0$,

$$(|\mathbf{n}|+1)^{2\sigma+1} \leqslant \bar{c}_0(\sigma)\,\rho^{-(2\sigma+1)}e^{\rho|\mathbf{n}|} \tag{6.46}$$

yields

$$|W| + \frac{|W-W'|}{|\bar{\alpha}-\bar{\alpha}'|} < \frac{c_0(\sigma)\,e^{\rho|\mathbf{n}|}}{\gamma_{k+1}^2\rho^{2\sigma+1}}\left(|V| + \frac{|V-V'|}{|\bar{\alpha}-\bar{\alpha}'|}\right) \tag{6.47}$$

Inserting this into the definition of the norm, we obtain finally the estimate (6.22).

The estimate (6.24) follows immediately from (6.14).

Remark. In the next lemmas we will use the following properties of the norm (6.18):

$$\begin{aligned}\|A\|_{r-\rho,\Omega} &\leqslant \|A\|_{r,\Omega} \qquad (0\leqslant \rho<r)\\ \|A\|_{r,\Omega_{k+1}} &\leqslant \|A\|_{r,\Omega_k} \qquad (\Omega_{k+1}\subset\Omega_k)\\ \|AB\|_{r,\Omega} &\leqslant \|A\|_{r,\Omega}\,\|B\|_{r,\Omega}\\ \|[A,B]\|_{r,\Omega} &\leqslant 2\,\|A\|_{r,\Omega}\,\|B\|_{r,\Omega}\end{aligned} \tag{6.48}$$

The third inequality is proven in ref. 33 (Lemma 2.3). The space of infinite matrices endowed with this norm is a Banach algebra.

Proof of Lemma 6.3. The condition (6.25) implies that $\|W_k\|_k<M$, for some constant M. We will use the inequality

$$\begin{aligned}\|R_k-1\|_k \equiv \|e^{W_k}-1\|_k &= \left\| W_k \sum_{j=1}^{\infty} \frac{1}{j!}(W_k)^{j-1}\right\|_k\\ &\leqslant \|W_k\|_k\, e^{\|W_k\|_k} \leqslant e^M\,\|W_k\|_k\end{aligned} \tag{6.49}$$

We can write $S_k=R_kS_{k-1}$ and estimate the difference

$$\begin{aligned}\|S_k-S_{k-1}\|_k = \|(R_k-1)\,S_{k-1}\|_k &\leqslant \|(R_k-1)\|_k\,\|S_{k-1}\|_k\\ &\leqslant e^M\,\|W_k\|_k\,\|S_{k-1}\|_k\end{aligned} \tag{6.50}$$

The last factor is bounded by

$$\begin{aligned}\|S_{k-1}\|_k = \left\|\prod_{k'=1}^{k-1} R_{k'}\right\|_k &\leqslant \prod_{k'=1}^{k-1} \|R_{k'}-1+1\|_{k'}\\ &\leqslant \prod_{k'=1}^{k-1}(e^M\,\|W_{k'}\|_{k'}+1) \leqslant \exp\left\{\sum_{k'=1}^{k-1}\ln(e^M\,\|W_{k'}\|_{k'}+1)\right\}\\ &\leqslant \exp\left\{\sum_{k'=1}^{k-1} e^M\,\|W_{k'}\|_{k'}\right\} \leqslant \exp\{e^M a\}\end{aligned} \tag{6.51}$$

Since $\|W_{k'}\|_k \leqslant \|W_{k'}\|_{k'} \to 0$ as $k'\to\infty$, we obtain $\lim_{k\to\infty}\|S_k-S_{k-1}\|_k=0$, and thus the sequence $\{S_k\}$ converges to a unitary operator R.

Proof of Lemma 6.4. From (6.8) we can write

$$\begin{aligned}\|V_{k+1}\| &\leqslant 2\,\|W_{k+1}\|\cdot\|V_k\| + \frac{2^2}{2!}\,\|W_{k+1}\|^2\cdot\|V_k\| + \cdots \\ &= 2\,\|W_{k+1}\|\cdot\|V_k\| \sum_{j=0}^{\infty} 2^j\,\|W_{k+1}\|^j \frac{1}{(j+1)!}\end{aligned} \tag{6.52}$$

and thus

$$\begin{aligned}\|V_{k+1}\|_{k+1} &\leqslant 2\,\|W_{k+1}\|_{k+1}\,\|V_k\|_{k+1}\exp(2\,\|W_{k+1}\|_{k+1}) \\ &\leqslant 2\,\|W_{k+1}\|_{k+1}\,\|V_k\|_k\exp(2\,\|W_{k+1}\|_{k+1})\end{aligned} \tag{6.53}$$

Proof of Lemma 6.5. (i) Denoting $x_k = A_k\varepsilon_k$, the inequality (6.27) becomes

$$x_{k+1} \leqslant 2^\nu x_k^2 e^{x_k} \tag{6.54}$$

We proceed in two steps. First we show by induction that if a sequence $\{x_k\}$ satisfies (6.54) and $x_0 < b$ for some b satisfying the condition

$$2^\nu b e^b < 1 \tag{6.55}$$

then $x_k < b$ for all k: Assume $x_k < b$; then $x_{k+1} \leqslant 2^\nu x_k^2 e^{x_k} \leqslant 2^\nu b^2 e^b < b$. A possible choice of b that satisfies (6.55) is $b = 1/(2^\nu e)$.

In a second step we improve the estimate: the bound $x_k < b$ together with (6.54) implies that x_k satisfies

$$x_{k+1} \leqslant c x_k^2 \qquad \text{with} \quad c = 2^\nu e^b > 1 \tag{6.56}$$

which in turn implies $x_k \leqslant c^{-1}(cx_0)^{2^k}$. The condition for convergence is therefore $cx_0 < 1$, i.e., $x_0 < 1/c = 2^{-\nu}e^{-b}$. But the condition (6.55) that we imposed in the first step implies that $b < 2^{-\nu}e^{-b}$. Therefore, the condition $x_0 < b = 1/(2^\nu e)$, which translates to

$$\varepsilon_0 < 1/(B2^\nu e) \tag{6.57}$$

guarantees that there is a positive $\mu < 1$ such that

$$\varepsilon_k < c_4 2^{-\nu k}\mu^{2^k}, \qquad c_4 = \text{const} \tag{6.58}$$

and also (ii):

$$\|W_k\|_k < A_k\varepsilon_k < c_5\mu^{2^k}, \qquad c_5 = \text{const} \tag{6.59}$$

(iii) We have to impose a further condition on b such that $\|g_k\|_k \leqslant \sum_{k=0}^{\infty} \varepsilon_k < \omega_1 \zeta/4$. As we have seen, the condition $x_k \leqslant b$ implies that $\varepsilon_k \leqslant (b/B)\, 2^{-\nu k}$ and therefore

$$\sum_{k=0}^{\infty} \varepsilon_k \leqslant \frac{b}{B} \frac{2^\nu}{2^\nu - 1} \tag{6.60}$$

which is smaller than $\omega_1 \zeta/4$ if $b < B(2^\nu - 1)/2^\nu$. This leads to $\varepsilon_0 < \omega_1 \zeta (2^\nu - 1)/2^{\nu+2}$. This completes the proof of Theorem 6.

Remark. The proof of this theorem generalizes immediately to the case of N-level models. The theorem can be extended to the case in which the perturbation $V(\theta_1, \theta_2)$ is differentiable (C^4) but not analytic, by applying the methods developed in ref. 33.

APPENDIX

Proof of Theorem 4.3. We will show by contradiction that the conditions (4.25), (4.26), and (4.27a) [or (4.27b)] imply that Eq. (4.10) has no solution. Consider some $\mathbf{n} = (n_1, n_2) \in \mathbb{Z}^2$ from the subsequence $\mathbf{n}_k$, which enters the conditions (4.26), (4.27a). To simplify notations, we denote $m = n_1$ and $n = -n_2$, so that $\mathbf{n} = (m, -n)$. Let

$$\varepsilon = m\alpha - n = \frac{m\omega_1 - n\omega_2}{\omega_2} = \frac{\mathbf{n} \cdot \boldsymbol{\omega}}{\omega_2}$$

For the sake of definiteness we will assume that $m, n > 0$. Let us denote

$$t = \frac{\theta_1}{2\pi}, \qquad g(t) = -\varphi(2\pi t), \qquad \beta(t) = -[v(2\pi t) - \mu] \tag{A.1}$$

Then (4.10) becomes

$$g(t + \alpha) - g(t) = \beta(t) \mod 2\pi \tag{A.2}$$

where t is defined mod 1 and

$$\begin{aligned} \beta(t) &= \int_0^{T_2} f(\omega_1 t' + 2\pi t, \omega_2 t')\, dt' + \mu \\ &= (\omega_2)^{-1} \int_0^{2\pi} f(\alpha t' + 2\pi t, t')\, dt' + \mu \end{aligned} \tag{A.3}$$

We can include μ into $f(t, t')$ by multiplying it by $\mu/(2\pi\omega_2)$, and therefore we may assume that $\mu = 0$. Equation (A.2) implies that

$$g(t+\varepsilon) - g(t) = \sum_{k=0}^{m-1} \beta(t + k\alpha) \bmod 2\pi \tag{A.4}$$

Let us estimate the RHS of this equation. Let

$$C = \sup_{0 \leqslant t \leqslant 1} |\beta'(t)|$$

Then

$$\left|\beta(t+k\alpha) - \beta\left(t + k\frac{n}{m}\right)\right| \leqslant Ck\left|\alpha - \frac{n}{m}\right| \leqslant C|\varepsilon|$$

and we can write

$$\sum_{k=0}^{m-1} \beta(t+k\alpha) = \sum_{k=0}^{m-1} \beta\left(t + k\frac{n}{m}\right) + \delta(t) \tag{A.5}$$

where

$$|\delta(t)| \leqslant Cm|\varepsilon| \tag{A.6}$$

Further, since

$$\sum_{k=0}^{m-1} \beta\left(t + k\frac{m}{n}\right) = \sum_{k=0}^{m-1} \beta\left(t + \frac{k}{n}\right)$$

(it is just only a permutation of terms), we get from (A.4) and (A.5) that

$$g(t+\varepsilon) - g(t) = \sum_{k=0}^{m-1} \beta\left(t + \frac{k}{m}\right) + \delta(t) \bmod 2\pi \tag{A.7}$$

We define a new function

$$\gamma(t) = (\omega_2)^{-1} \int_0^{2\pi} f\left(\frac{n}{m} t' + 2\pi t, t'\right) dt' \tag{A.8}$$

Comparing with (A.3), its difference with β is bounded by

$$|\gamma(t) - \beta(t)| \leqslant C\frac{|\varepsilon|}{m}$$

and we can rewrite (A.7) as

$$g(t+\varepsilon) - g(t) = \sum_{k=0}^{m-1} \gamma\left(t + \frac{k}{m}\right) + \delta(t) \bmod 2\pi \tag{A.9}$$

where $\delta(t)$ is a different function but it still satisfies estimate (A.6) (with a new constant C).

Let us expand $\gamma(t)$ in the Fourier series

$$\gamma(t)=\sum_{j=-\infty}^{\infty}\tilde{\gamma}(j)\exp(2\pi ijt) \tag{A.10}$$

and substitute it into (A.9). Since

$$\sum_{k=0}^{m-1}\exp\left(2\pi ij\frac{k}{m}\right)=\begin{cases}0, & j\neq lm\\ m, & j=lm\end{cases}$$

where $l\in\mathbb{Z}$, we get that

$$\sum_{k=0}^{m-1}\gamma\left(t+\frac{k}{m}\right)=m\sum_{l=-\infty}^{\infty}\tilde{\gamma}(lm)\exp(2\pi ilmt) \tag{A.11}$$

Let us compute $\tilde{\gamma}(lm)$:

$$\begin{aligned}
\tilde{\gamma}(lm)&=\int_0^1\gamma(t)\exp(-2\pi ilmt)\,dt\\
&=(\omega_2)^{-1}\int_0^1\int_0^{2\pi}f\left(2\pi t+\frac{n}{m}t',t'\right)\exp(-2\pi ilmt)\,dt\,dt'\\
&=(2\pi\omega_2)^{-1}\int_0^{2\pi}\int_0^{2\pi}f(t,t')\exp\left[-ilm\left(t-\frac{n}{m}t'\right)\right]dt\,dt'\\
&=T_2\tilde{f}(lm,-ln)
\end{aligned}$$

Substituting it into (A.9), (A.11), we get that

$$\begin{aligned}
g(t+\varepsilon)-g(t)&=T_2m\sum_{l=-\infty}^{\infty}\tilde{f}(lm,-ln)\exp(2\pi ilmt)+\delta(t)\\
&=I_0+h_0(t)+\delta(t)\bmod 2\pi
\end{aligned} \tag{A.12}$$

with $I_0=T_2m\tilde{f}(0,0)$ and

$$h_0(t)=T_2m\sum_{l\neq 0}\tilde{f}(lm,-ln)\exp(2\pi ilmt)$$

Since Eq. (A.12) is defined mod 2π, we may change I_0 by $2\pi k$ and assume that $-\pi<I_0\leqslant\pi$. For the sake of definiteness we will assume that

$$0\leqslant I_0\leqslant\pi \tag{A.13}$$

The function $f(t_1, t_2)$ is real; therefore,

$$\tilde{f}(-lm, ln) = \tilde{f}^*(lm, -ln)$$

hence

$$\sum_{l \neq 0} \tilde{f}(lm, -ln) \exp(2\pi i lmt) = 2 \sum_{l=1}^{\infty} |\tilde{f}(lm, -ln)| \cos[2\pi lm(t - \Delta_{lm})]$$

Let us denote

$$q(t) = \sum_{l=1}^{\infty} |\tilde{f}(lm, -ln)| \cos[2\pi lm(t - \Delta_{lm})]$$

where $\Delta_{lm} = -\arg(\tilde{f}(lm, -ln))$. We do not note explicitly the dependence on n, since it is not relevant for the argument. By the conditions of the theorem, either (4.27a) or (4.27b) holds. Let us consider first the case when (4.27a) holds. It ensures that for the values $t = \Delta_{1m}$

$$\begin{aligned} q(\Delta_{1m}) &= |\tilde{f}(m, -n)| + \sum_{l=2}^{\infty} |\tilde{f}(lm, -ln)| \cos[2\pi lm(\Delta_{1m} - \Delta_{lm})] \\ &\geqslant |\tilde{f}(m, -n)| - \sum_{l=2}^{\infty} |\tilde{f}(lm, -ln)| \geqslant \varepsilon_0 \, |\tilde{f}(m, -n)| \end{aligned}$$

Moreover, there exists a segment $[a_0, b_0]$ around the point $t_0 = \Delta_m$ such that

$$2\,|\tilde{f}(m, -n)| \geqslant q(t) \geqslant \frac{\varepsilon_0}{2}\,|\tilde{f}(m, -n)| \tag{A.14}$$

when $t \in [a_0, b_0]$, and

$$|b_0 - a_0| \geqslant \frac{\zeta_0}{m}$$

where $\zeta_0 > 0$ does not depend on m. One can construct similar segments $[a_j, b_j]$ around each point $t_j = \Delta_{1m} + j/m$, $j = 0, 1, \ldots, m-1$, where $\cos[2\pi m(t - \Delta_{1m})]$ is equal to 1. Thus, we have a set of nonintersecting segments $[a_j, b_j]$, $j = 0, 1, \ldots, m-1$, such that for $t \in \bigcup_{j=0}^{m-1} [a_j, b_j]$ the estimate (A.14) is valid and

$$|b_j - a_j| \geqslant \frac{\zeta_0}{m} \tag{A.15}$$

Returning to (A.12), we have that

$$g(t+\varepsilon)-g(t)=I_0+h_0(t)+\delta(t) \bmod 2\pi$$

where $h_0(t)=2T_2mq(t)$ satisfies for $t\in\bigcup_{j=0}^{m-1}[a_j, b_j]$ the estimate

$$Cm\,|\tilde{f}(m,\,-n)|>h_0(t)>C^{-1}m\,|\tilde{f}(m,\,-n)|$$

with some C not depending on m. Since by (A.6)

$$|\delta(t)|<C_0m\,|\mathbf{n}\cdot\boldsymbol{\omega}|$$

then by the condition (4.26) a similar estimate holds, starting with some $m=m_0$, also for $h(t)=h_0(t)+\delta(t)$, i.e.,

$$g(t+\varepsilon)-g(t)=I_0+h(t) \bmod 2\pi \tag{A.16}$$

where for $t\in\bigcup_{j=0}^{m-1}[a_j, b_j]$,

$$Cm\,|\tilde{f}(m,\,-n)|>h(t)>C^{-1}m\,|\tilde{f}(m,\,-n)| \tag{A.17}$$

with C not depending on m. For what follows it is useful to notice that the condition (4.25) implies that

$$\lim_{m\to\infty} mf(m,\,-n)=0$$

If the function $g(t)$ were differentiable, we would get from (A.16) and (A.17) that

$$C_0\,|\mathbf{n}\cdot\boldsymbol{\omega}|>\sup_{0\leqslant t\leqslant 1}|g'(t)|\cdot|\varepsilon|>C^{-1}m\,|\tilde{f}(m,\,-n)|=C^{-1}m\,|\tilde{f}(\mathbf{n})|$$

which contradicts the condition (4.26). However, we can only assume that $g(t)$ is measurable, so we will use more sophisticated considerations.

Since Eq. (A.16) is mod 2π, we may assume that

$$0\leqslant g(t)<2\pi$$

Let us represent $g(t)$ as

$$g(t)=g_0(t)+g_1(t), \qquad 0\leqslant g_0(t),\ g_1(t)<2\pi$$

where $g_1(t)\in C^1$ and

$$\int_0^1 |g_0(t)|\,dt<\delta \tag{A.18}$$

where $\delta > 0$ will be chosen later. Then there exist $C_0 = C_0(\delta) > 0$ and $m_0 = m_0(\delta)$ such that

$$|g_1(t+\varepsilon) - g_1(t)| < C_0 \varepsilon < \tfrac{1}{2} C^{-1} m \, |\tilde{f}(m, -n)|$$

for $m > m_0(\delta)$, so again we have for $g_0(t)$ that

$$g_0(t+\varepsilon) - g_0(t) = I_0 + h(t) \bmod 2\pi \tag{A.19}$$

with some new $h(t)$ which satisfies for $m > m_0(\delta)$ the estimate (A.17) as well with a new constant $2C$ instead of C. To simplify notations, we will denote this new constant again by C. For what follows it is important that C does not depend on δ and on the splitting of $g(t)$ into $g_0(t) + g_1(t)$.

We use now the following idea. Consider on the segment $[a_0, b_0]$ the sequence of points

$$c_k(\varepsilon') = a_0 + \varepsilon' + (k-1)\,\varepsilon, \qquad k = 1, 2, ..., K = \left[\frac{|b_0 - a_0|}{\varepsilon}\right]$$

with some arbitrary ε', $0 \leqslant \varepsilon' \leqslant \varepsilon$ (the square brackets denote the integer part). We are interested in those k's for which

$$\frac{\pi}{4} < g_0(c_k(\varepsilon')) < \frac{7\pi}{4} \tag{A.20}$$

Denote $A(\varepsilon')$ the set of such k's. Let $|A(\varepsilon')|$ be the number of elements in $A(\varepsilon')$. We will show that

$$|A(\varepsilon')| > \gamma K \tag{A.21}$$

where $\gamma > 0$ does not depend on m and ε': Denote by $A^c(\varepsilon') = \{1, 2, ..., K\} \backslash A(\varepsilon')$ the complement of $A(\varepsilon')$. Then it follows from Eq. (A.19) and the estimate (A.17) that $A(\varepsilon')$ and $A^c(\varepsilon')$ consist of alternating intervals of subsequent integers such that the length of every interval of $k \in A(\varepsilon')$ is not less than

$$l_0 = \left[\frac{7\pi/4 - \pi/4}{I_0 + Cm\,|f(m, -n)|}\right] = \left[\frac{3\pi}{2(I_0 + Cm\,|f(m, -n)|)}\right]$$

and the length of every interval of $k \in A^c(\varepsilon')$ is not larger than

$$l_1 = \left[\frac{\pi}{2(I_0 + C^{-1} m\,|f(m, -n)|)}\right] + 1$$

Hence

$$|A(\varepsilon')| \geqslant l_0 L$$

where $L > 1$ is the number of intervals in $A(\varepsilon')$ and

$$|A^c(\varepsilon')| \leqslant l_1(L+1)$$

so

$$\frac{|A(\varepsilon')|}{|A^c(\varepsilon')|} \geqslant \frac{l_0}{l_1} \frac{L}{L+1} > \gamma_0 > 0$$

with γ_0 not depending on m and ε', so

$$|A(\varepsilon')| \geqslant \frac{\gamma_0}{1+\gamma_0} (|A(\varepsilon')| + |A^c(\varepsilon')|) = \frac{\gamma_0}{1+\gamma_0} K$$

which proves (A.21).

Since γ does not depend on ε', we get from (A.19) that the Lebesgue measure of the set

$$A = \left\{ t \in [a_0, b_0] \,\middle|\, \frac{\pi}{4} \leqslant g_0(t) \leqslant \frac{7\pi}{4} \right\}$$

is not less than $\gamma\, |b_0 - a_0|$, so

$$\int_{a_0}^{b_0} |g_0(t)|\, dt \geqslant \int_A |g_0(t)|\, dt \geqslant \frac{\pi}{4} \int_A dt \geqslant \frac{\pi}{4} \gamma\, |b_0 - a_0| \geqslant \frac{\gamma}{2} |b_0 - a_0|$$

A similar estimate can be proven for any interval $[a_j, b_j]$:

$$\int_{a_j}^{b_j} |g_0(t)|\, dt \geqslant \frac{\gamma}{2} |b_j - a_j|, \qquad j = 0, 1, ..., m-1 \tag{A.22}$$

By (A.15) it implies that

$$\int_{a_j}^{b_j} |g_0(t)|\, dt \geqslant \frac{\gamma_0}{m}$$

where $\gamma_0 > 0$ does not depend on m. Hence

$$\int_0^1 |g_0(t)|\, dt \geqslant \sum_{j=1}^{m-1} \int_{a_j}^{b_j} |g_0(t)|\, dt \geqslant \gamma_0 > 0$$

Remark that γ_0 can be chosen independently of the splitting $g(t)$ into $g_0(t)+g_1(t)$, so the last estimate contradicts the one of Eq. (A.18) if we choose $\delta<\gamma_0$. This contradiction proves Theorem 4.3 in the case when (4.27a) holds for some sequence $\mathbf{n}=(m,-n)$ with $m\geqslant m_0(\delta)$.

Let us consider now the case when (4.27b) holds for large m. Again the idea is to construct a number of intervals $[a_j, b_j]$ for which (A.14), (A.15) are valid. We have

$$\|q(t)\|_{L^2}=\left\|\sum_{l=1}^{\infty}|\tilde{f}(lm,-ln)|\cos[2\pi lm(t-\Delta_{lm})]\right\|_{L^2}$$

$$\geqslant\frac{1}{\sqrt{2}}|f(m,-n)| \tag{A.23}$$

and by (4.27b)

$$\|q(t)\|_C\leqslant\sum_{l=1}^{\infty}|\tilde{f}(lm,-ln)|\leqslant\sum_{l=1}^{\infty}l\,|\tilde{f}(lm,-ln)|\leqslant\frac{1+\varepsilon_0}{\varepsilon_0}|\tilde{f}(m,-n)| \tag{A.24}$$

and

$$\left\|\frac{dq(t)}{dt}\right\|_C\leqslant m\sum_{l=1}^{\infty}l\,|\tilde{f}(lm,-ln)|\leqslant m\frac{1+\varepsilon_0}{\varepsilon_0}|\tilde{f}(m,-n)| \tag{A.25}$$

Besides,

$$\int_0^1 q(t)\,dt=0 \tag{A.26}$$

It follows from (A.23) and (A.24) that

$$\frac{1}{2}|\tilde{f}(m,-n)|^2\leqslant\|q(t)\|_{L^2}^2\leqslant\|q(t)\|_C\,\|q(t)\|_{L^1}\leqslant\frac{1+\varepsilon_0}{\varepsilon_0}|\tilde{f}(m,-n)|\;\|q(t)\|_{L^1}$$

so that

$$\|q(t)\|_{L^1}\geqslant\frac{\varepsilon_0}{2(1+\varepsilon_0)}|\tilde{f}(m,-n)| \tag{A.27}$$

Denote

$$A_\delta=\{t\,|\,q(t)>\delta\}$$

It follows from (A.26) and (A.27) that

$$\int_{A_0} q(t)\,dt = \frac{1}{2}\,\|q(t)\|_{L^1} \geqslant \frac{\varepsilon_0}{4(1+\varepsilon_0)}\,|\tilde{f}(m,\,-n)| \tag{A.28}$$

Now, for every $\delta > 0$,

$$\begin{aligned}\int_{A_0} q(t)\,dt &= \int_{A_\delta} q(t)\,dt + \int_{A_0\backslash A_\delta} q(t)\,dt \\ &\leqslant \|q(t)\|_C\,\mathscr{L}(A_\delta) + \delta\mathscr{L}(A_0\backslash A_\delta) \\ &\leqslant \frac{1+\varepsilon_0}{\varepsilon_0}\,|\tilde{f}(m,\,-n)|\;\mathscr{L}(A_\delta) + \delta\end{aligned}$$

where $\mathscr{L}$ denotes the Lebesgue measure. Combining this with (A.28), we get that

$$\frac{\varepsilon_0}{4(1+\varepsilon_0)}\,|\tilde{f}(m,\,-n)| \leqslant \int_{A_0} q(t)\,dt \leqslant \frac{1+\varepsilon_0}{\varepsilon_0}\,|\tilde{f}(m,\,-n)|\;\mathscr{L}(A_\delta) + \delta$$

Therefore for

$$\delta = \frac{\varepsilon_0}{8(1+\varepsilon_0)}\,|\tilde{f}(m,\,-n)|$$

we have the estimate

$$\mathscr{L}(A_\delta) \geqslant \frac{\varepsilon_0^2}{8(1+\varepsilon_0)^2} \equiv \lambda > 0 \tag{A.29}$$

Let $t_0 \in A_\delta$, so that $q(t_0) > \delta$. Then, for

$$|t - t_0| < \frac{\varepsilon_0^2}{16m(1+\varepsilon_0)^2} \equiv \frac{\mu}{m}$$

we get by (A.25) that

$$q(t) - q(t_0) \leqslant m\,\frac{1+\varepsilon_0}{\varepsilon_0}\,|\tilde{f}(m,\,-n)| \cdot |t - t_0| \leqslant \frac{|\tilde{f}(m,\,-n)|\;\varepsilon_0}{16(1+\varepsilon_0)} = \frac{\delta}{2}$$

Hence, for $|t - t_0| \leqslant \mu/m$,

$$q(t) \geqslant \frac{\delta}{2} = \frac{\varepsilon_0}{16(1+\varepsilon_0)}\,|\tilde{f}(m,\,-n)| \tag{A.30}$$

Because of (A.29) one can find $N = [(\lambda/(2\mu))\, m]$ points $t_0, \ldots, t_{N-1} \in A_\delta$ such that their μ/m-neighborhoods do not intersect, and for every t from those neighborhoods (A.30) holds. This implies that the bounds (A.14) and (A.15) are also valid in this case and the remainder of the proof can be repeated without change. This completes the proof of Theorem 4.3.

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